LESSON 22 - STUDY GUIDE

Abstract. In this lesson we will start studying the problem of convergence in $L^p(\mathbb{T})$ norm of the partial sums of Fourier series. We will begin by seeing that it is equivalent to the uniform boundedness of the operator norms of the partial sums. Then, introducing the important connection of Fourier series with the theory of complex analysis on the unit disk, we will study the problem of determining holomorphic and harmonic extensions in the interior of the disk, from given boundary values. We will finish by introducing the conjugation operator and show that convergence of Fourier series in $L^p(\mathbb{T})$ norm is equivalent to conjugation being well defined in $L^p(\mathbb{T})$.

1. Convergence in norm, harmonic functions on the unit disk and the conjugation operator.

Study material: This lesson is a very expanded version of section **1 - Convergence in Norm** from chapter **II - The Convergence of Fourier Series**, corresponding to pgs. 46–49 in the second edition [3] and pgs. 66–70 in the third edition [4] of Katznelson's book. The middle presentation of the study of holomorphic and harmonic functions on the unit disk in complex analysis, to motivate and introduce the conjugation operator, is essentially my own and is not in Katznelson in this level of detail. My main reference for this subject are the first chapters of the beautiful little book by Kenneth Hoffman [2] about the theory of analytic and harmonic functions on the unit disk, so called Hardy spaces, which we will keep getting back to in further lectures.

We will now start looking at the issue of the convergence of the partial sums of the Fourier series in $L^p(\mathbb{T})$ norm. Clearly, we already have some results in that direction. The most significant of which is the convergence of Fourier series in the $L^2(\mathbb{T})$ norm, obtained in Lesson 16, as a consequence of the Hilbert space structure in that case. In the opposite direction we have the $L^\infty(\mathbb{T})$ situation where convergence in norm of the partial sums of the Fourier series would correspond to uniform convergence and therefore the limit would have to be continuous, which is not the case with arbitrary $L^\infty(\mathbb{T})$ functions. The situation does not improve if we restrict to $C(\mathbb{T})$ because we have seen, in Lesson 19, the dramatic examples of Fourier series of continuous functions that diverge pointwise in uncountable dense sets. So, while convergence in norm holds in $L^2(\mathbb{T})$ it does not in $L^\infty(\mathbb{T})$ and one cannot interpolate between a positive and a negative result. We will now see that at the other end of the $L^p(\mathbb{T})$ spaces, in $L^1(\mathbb{T})$, convergence does not hold either.

We start with the following theorem.

Theorem 1.1. *For every $f \in L^p(\mathbb{T})$, $1 \le p < \infty$, the partial sums $S_N[f](t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$ of the Fourier series converge to f in the $L^p(\mathbb{T})$ norm, if and only if the partial sum operators $S_N : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ are uniformly bounded in $N \in \mathbb{N}$, i.e. if there exists a constant $C > 0$ such that*

$$\|S_N[f]\|_{L^p(\mathbb{T})} \le C\|f\|_{L^p(\mathbb{T})}, \tag{1.1}$$

for all $f \in L^p(\mathbb{T})$ and $N \in \mathbb{N}$.

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Proof. If the partial sums converge in norm, then for any $f \in L^p(\mathbb{T})$ the sequence of partial sums is bounded in $L^p(\mathbb{T})$, for it is a convergent sequence. Therefore, there exists a constant C_f, depending on f, such that

$$\|S_N[f]\|_{L^p(\mathbb{T})} \leq C_f.$$

But then, from the Banach-Steinhaus theorem of functional analysis, that we recalled in Lesson 19, this implies that the family of continuous operators S_N is uniformly bounded

$$\|S_N\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \leq C,$$

for some $C > 0$ and all $N \in \mathbb{N}$. An this is the same as (1.1).

In the opposite direction, if (1.1) holds, then for arbitrarily small ε one can pick a trigonometric polynomial P such that $\|P - f\|_{L^p} < \varepsilon/(C+1)$ to do

$$\|S_N[f] - f\|_{L^p(\mathbb{T})} \leq \|S_N[f] - p\|_{L^p(\mathbb{T})} + \|P - f\|_{L^p(\mathbb{T})},$$

and taking $N > \text{degree}(P)$, we get $P = S_N[P]$ from which

$$\begin{aligned} \|S_N[f] - f\|_{L^p(\mathbb{T})} &\leq \|S_N[f] - S_N[P]\|_{L^p(\mathbb{T})} + \|P - f\|_{L^p(\mathbb{T})} \\ &= \|S_N[f - P]\|_{L^p(\mathbb{T})} + \|P - f\|_{L^p(\mathbb{T})} \\ &\leq C\|P - f\|_{L^p(\mathbb{T})} + \|P - f\|_{L^p(\mathbb{T})} \leq \varepsilon, \end{aligned}$$

concluding the proof □

In $L^1(\mathbb{T})$, the operator norms of the partial sums are, quite unsurprisingly, the Lebesgue constants. In fact

$$\|S_N[f]\|_{L^1(\mathbb{T})} = \|D_N * f\|_{L^1(\mathbb{T})} \leq \|D_N\|_{L^1(\mathbb{T})}\|f\|_{L^1(\mathbb{T})},$$

so that $\|S_N\|_{L^1(\mathbb{T})\to L^1(\mathbb{T})} \leq \|D_N\|_{L^1(\mathbb{T})}$. And, using any approximate identity as f, for example the Fejér kernel, one gets

$$\|S_N[K_j]\|_{L^1(\mathbb{T})} = \|D_N * K_j\|_{L^1(\mathbb{T})} \leq \|S_N\|_{L^1(\mathbb{T})\to L^1(\mathbb{T})},$$

because $\|K_j\|_{L^1(\mathbb{T})} = 1$. But making $j \to \infty$ on the left hand side, we get $\|D_N * K_j\|_{L^1(\mathbb{T})} \to \|D_N\|_{L^1(\mathbb{T})}$ and thus $\|D_N\|_{L^1(\mathbb{T})} \leq \|S_N\|_{L^1(\mathbb{T})\to L^1(\mathbb{T})}$ concluding finally that

$$\|D_N\|_{L^1(\mathbb{T})} = \|S_N\|_{L^1(\mathbb{T})\to L^1(\mathbb{T})}.$$

So, from the previous theorem, and the fact that the Lebesgue constants grow logarithmically in N we obtain the following corollary.

Corollary 1.2. *There exist functions $f \in L^1(\mathbb{T})$ for which their Fourier series do not converge in the $L^1(\mathbb{T})$ norm.*

So $L^1(\mathbb{T})$ is quite pathological, for we have examples of functions whose Fourier series do not converge in norm as well as functions whose Fourier series diverge pointwise everywhere, as mentioned in Lesson 20. In terms of convergence of the partial sums of Fourier series things cannot be much worse than that.

To try to investigate what happens for the remaining $L^p(\mathbb{T})$ spaces, $1 < p < \infty$, it is not easy to determine the operator norms of the partial sums. It turns out that convergence in norm can also be related to the definition of a particular Fourier multiplier operator, which surprisingly arises from looking at Fourier series as boundary values of harmonic functions on the unit complex disk.

Let us denote by $D = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk in the complex plane, and $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ its closure. $\mathbb{T}$ can then be isomorphically identified with the the boundary of the disk $\partial D = \mathbb{S}^1$ through $t \mapsto e^{it}$ and functions on $\mathbb{T}$ identified this way with functions on $\partial D = \mathbb{S}^1$. We will denote their values by either $f(t)$ or the pullback $f(e^{it})$, depending on the context, whether we are thinking of $\mathbb{T}$ or ∂D.

Recalling now a bit of elementary complex analysis, and imagining that $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function on an open set Ω that contains $\overline{D}$, we can expand f in Taylor series centered at the origin with a radius of convergence strictly larger than one. So that, for $|z| \leq 1$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

with the convergence holding absolutely and uniformly for all $z \in \overline{D}$. If we write $z = re^{it}$ in polar coordinates, then, for $|z| = r \leq 1$ we have

$$f(z) = f(re^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{int},$$

where $a_n = \frac{f^{(n)}(0)}{n!}$ are the coefficients of the Taylor series. By changing the complex analysis point of view just slightly, we can imagine that $f(z) = f(re^{it}) = f_r(t)$ is a one-parameter family of functions of the angle $t \in \mathbb{T}$, where the parameter is the radius $r \leq 1$. Therefore, not only at the boundary of the disk itself, but also for every fixed radius, $f(z) = f_r(t)$ can always be interpreted as one of our usual functions defined on the circle $\mathbb{T}$. And due to the absolute and uniform convergence of the Taylor series, we conclude that, at fixed r, the Taylor series actually yields the Fourier series of f_r, where

$$\widehat{f_r}(n) = \begin{cases} a_n r^n & \text{for} \quad n \geq 0 \\ 0 \quad \text{for} & n < 0. \end{cases}$$

In particular, the Taylor series coefficients $a_n = \frac{f^{(n)}(0)}{n!}$ are the Fourier coefficients of f at the boundary of the disk ∂D, at radius $r = 1$, for the nonnegative frequencies,

$$f(e^{it}) = f_1(t) = \sum_{n=0}^{\infty} a_n e^{int} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} e^{int}.$$

And this then implies that, for $r < 1$, f_r can be interpreted as the result of a Fourier multiplier operator applied to the boundary function f_1, with multiplier coefficients r^n, for $n \geq 0$, and 0, for $n < 0$. Or, equivalently, the convolution of $f_1(t) = f(e^{it})$ with the kernel

$$C_r(t) = \sum_{n=0}^{\infty} r^n e^{int} = \frac{1}{1 - re^{it}},$$

so that, for $r < 1$,

$$\begin{aligned} f(z) = f(re^{it}) &= f_r(t) = f_1 * C_r(t) = \frac{1}{2\pi} \int_{\mathbb{T}} f_1(s) C_r(t-s) ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(s) \frac{1}{1 - re^{i(t-s)}} ds = \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(e^{is}) \frac{ie^{is}}{e^{is} - re^{it}} ds \\ &= \frac{1}{2\pi i} \oint_{|w|=1} \frac{f(w)}{w - z} dw. \end{aligned} \tag{1.2}$$

So we have concluded the Cauchy integral formula of complex analysis for the values of a holomorphic function in D from its boundary values, by a Fourier analysis interpretation of the Taylor series at every fixed radius, as a Fourier multiplier operator that corresponds to the convolution of the function at the boundary $r = 1$ with the kernel C_r. Accordingly, this kernel is called the *Cauchy* kernel.

Likewise, we can do a very analogous reasoning as above but now for harmonic functions. Recall that a function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$ is said to be harmonic if it is twice continuously differentiable on the open set Ω and satisfies Laplace's equation there

$$\Delta f = 0 \Leftrightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

which is equivalent to its real and imaginary parts $f = u + iv$ being harmonic too. In particular, every holomorphic function on an open set is harmonic, because it is an immediate consequence of the Cauchy-Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \end{cases}$$

and the fact that holomorphic functions are infinitely differentiable, that we have, by differentiating the Cauchy-Riemann equations,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Now, recall the important converse result that, if a real function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is harmonic and the domain Ω is simply connected, then there exists another real harmonic function $v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, unique up to an additive constant, such that $u + iv$ is holomorphic on Ω. This function v is called the *harmonic conjugate* of u.

So if, as before, Ω is open and contains $\overline{D}$ and we now start with $u : \Omega \to \mathbb{R}$ harmonic, then there exists a harmonic conjugate v on an open set that contains $\overline{D}$, unique if we demand that $v(0) = 0$. Then, $f = u + iv$ is holomorphic on $\overline{D}$ so that it is given by a Taylor series centered at the origin

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{int}, \tag{1.3}$$

with $f(0) = u(0) = a_0 \in \mathbb{R}$, uniformly and absolutely convergent for $|z| \le 1$, and therefore

$$u(z) = \mathrm{Re}(f)(z) = \frac{f(z) + \overline{f(z)}}{2} = \frac{\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{a_n} \bar{z}^n}{2}, \tag{1.4}$$

while

$$v(z) = \mathrm{Im}(f)(z) = \frac{f(z) - \overline{f(z)}}{2i} = \frac{\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} \overline{a_n} \bar{z}^n}{2i}. \tag{1.5}$$

Expanding (1.4) in polar coordinates $z = re^{it}$, to turn the Taylor series again into a Fourier series for fixed radius r, we get

$$\begin{aligned} u(z) = u(re^{it}) = u_r(t) &= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} r^n e^{int} + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{2} r^n e^{-int} \\ &= \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{int}, \end{aligned}$$

with

$$c_n = \begin{cases} a_n/2 & \text{for} \quad n > 0, \\ a_0 & \text{for} \quad n = 0, \\ \overline{a_{-n}}/2 & \text{for} \quad n < 0, \end{cases}$$

where these coefficients are now the Fourier coefficients of u at $r = 1$, $\widehat{u_1}(n) = c_n$ for all $n \in \mathbb{Z}$, which satisfy the relation $\widehat{u_1}(n) = c_n = \overline{c_{-n}} = \overline{\widehat{u_1}(-n)}$, in accordance with the fact that u_1 is a real function.

At this point, like we did above for holomorphic functions, we can now interpret the Fourier series expansion of a harmonic function for $r < 1$, with Fourier coefficients given by $\widehat{u_r}(n) = c_n r^{|n|}$ for all $n \in \mathbb{Z}$, as the Fourier multiplier operator, with multipliers $r^{|n|}$, acting on the function u_1 on $\mathbb{T}$, the boundary value of u at ∂D. So that, in analogy with (1.2), for $|z| = r < 1$ we have

$$u(z) = u(re^{it}) = u_r(t) = u_1 * P_r(t) = \frac{1}{2\pi}\int_{\mathbb{T}} u_1(s)P_r(t-s)ds,$$

with P_r the convolution kernel associated to the Fourier multipliers $r^{|n|}$,

$$\begin{aligned} P_r(t) &= \sum_{n=-\infty}^{\infty} r^{|n|}e^{int} = 1 + 2\sum_{n=1}^{\infty} r^n \cos(nt) \\ &= \mathrm{Re}\left(1 + 2\sum_{n=1}^{\infty} z^n\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right) \\ &= \mathrm{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right) = \frac{1-r^2}{1-2r\cos t + r^2}. \end{aligned} \tag{1.6}$$

And this is none other than the Poisson kernel that we have encountered before, associated to Abel's summability method. So, while for holomorphic functions on the closed disk, whose Fourier series only have nonzero coefficients for nonnegative frequencies, their interior point values are given by the convolution of the boundary function with the Cauchy kernel, corresponding then to Cauchy's integral formula of complex analysis, for harmonic functions their Fourier series in general have nonzero coefficients for all frequencies, and their interior point values are given by the convolution of the boundary function with the Poisson kernel,

$$u(z) = u_r(t) = u_1 * P_r(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(s)\frac{1-r^2}{1-2r\cos(t-s)+r^2}ds. \tag{1.7}$$

This is the important Poisson integral formula, also frequently taught in complex analysis courses, for obtaining harmonic functions on D from their boundary values on ∂D.

As a side observation, note that the Poisson integral formula clearly also holds for complex harmonic functions, which happens, in particular, if the harmonic function is holomorphic. So we actually have both convolution formulas,

$$f(z) = f_r(t) = f_1 * C_r(t) = f_1 * P_r(t)$$

as equivalent methods for obtaining the values of a holomorphic function on D from its boundary values. This is not surprising by looking at the Fourier series representation of these operators, as they both have the same Fourier multipliers r^n for $n \geq 0$, and holomorphic functions have boundary values with Fourier coefficients $\widehat{f_1}(n) = 0$ for $n < 0$, as we saw before. So the Fourier multipliers for $n < 0$ really do not matter in this case and any other Fourier multiplier operator with the same r^n multipliers for $n \geq 0$, whatever its values were for $n < 0$, would do the same job of reconstructing the holomorphic function on D from its values on ∂D.

Returning to the main idea that we are pursuing, we can also obtain in a similar fashion both the holomorphic function f in (1.3) as well as the harmonic conjugate v in (1.5), from the boundary values u_1 of the harmonic function u, by using appropriate kernels. Observing from (1.6) that $P_r(t) = \mathrm{Re}\left(\frac{1+z}{1-z}\right) = \mathrm{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right)$ and that $\mathrm{Re}(f(z)) = u(z) = u_r(t) = u_1 * P_r = u_1 * \mathrm{Re}\left(\frac{1+z}{1-z}\right)$, if we now denote by $H(z) =$

$H_r(t)$ the function given, for $r<1$, by

$$\begin{aligned} H(z)=H_r(t) &= 1+2\sum_{n=1}^{\infty} z^n = 1+2\sum_{n=1}^{\infty} r^n e^{int} \\ &= \frac{1+z}{1-z} = \frac{1+2i\mathrm{Im}(z)-|z|^2}{|1-z|^2} = \frac{1+2i\,\mathrm{Im}(z)-|z|^2}{|1-z|^2} \\ &= \frac{1+re^{it}}{1-re^{it}} = \frac{1+2i\sin t - r^2}{1-2r\cos t + r^2} \end{aligned}$$

this analytic function is naturally called the *holomorphic Poisson kernel* and it allows us to obtain the unique holomorphic f in the interior of the unit disk that has zero imaginary part at the origin, from the values on ∂D of its real part, the harmonic function u, by the Fourier multiplier operator corresponding to the Taylor series in (1.3)

$$f(z)=f(re^{it})=f_r(t)=c_0+2\sum_{n=1}^{\infty} c_n r^n e^{int} = \widehat{u_1}(0)+2\sum_{n=1}^{\infty}\widehat{u_1}(n) r^n e^{int},$$

with zero Fourier coefficients for the negative frequencies, that we now understand to be the case for analytic functions. Or equivalently, by the convolution on the circle $\mathbb{T}$,

$$\begin{aligned} f(z)=f(re^{it})=f_r(t)=u_1 * H_r(t) &= \frac{1}{2\pi}\int_{\mathbb{T}} u_1(s)H_r(t-s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(s)\frac{1+2i\sin(t-s)-r^2}{1-2r\cos(t-s)+r^2}ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(s)\frac{1+re^{i(t-s)}}{1-re^{i(t-s)}}ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(s)\frac{e^{is}+re^{it}}{e^{is}-re^{it}}ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{is})\frac{e^{is}+z}{e^{is}-z}ds = \frac{1}{2\pi i}\int_{-\pi}^{\pi} u(e^{is})\frac{e^{is}+z}{e^{is}-z}\frac{ie^{is}}{e^{is}}ds \\ &= \frac{1}{2\pi i}\oint_{|w|=1}\frac{u(w)}{w}\frac{w+z}{w-z}dw. \end{aligned}$$

Of course the difference between this formula and (1.2), corresponding to Cauchy's integral formula, or the convolution with the Cauchy kernel, is that here we obtain the values of the holomorphic function f on D from the values at $r=1$ of just its real part u, whereas there we obtained f from its own values on the boundary.

Finally, we can still obtain the harmonic conjugate v also from the boundary values of u, this time by the convolution with the *conjugate Poisson kernel*

$$\begin{aligned} Q_r(t) &= -i\sum_{n=-\infty}^{\infty}\mathrm{sgn}(n) r^{|n|}e^{int} = \mathrm{Im}\left(2\sum_{n=1}^{\infty} z^n\right) \\ &= \mathrm{Im}\left(\frac{1+z}{1-z}\right) = \mathrm{Im}(H_r(t)) \\ &= \frac{2\sin t}{1-2r\cos t+r^2}, \end{aligned} \tag{1.8}$$

where $\mathrm{sgn}(0)=0$ and $\mathrm{sgn}(n)=\pm 1$ depending on whether $n>0$ or $n<0$. Therefore we can rewrite (1.5) as the Fourier multiplier operator

$$v(z)=v_r(t)=-i\sum_{n=-\infty}^{\infty}\mathrm{sgn}(n)c_n r^{|n|}e^{int} = -i\sum_{n=-\infty}^{\infty}\mathrm{sgn}(n)\widehat{u_1}(n) r^{|n|}e^{int}$$

or, equivalently, v can be obtained for $r<1$ by the convolution on $\mathbb{T}$,

$$v(z)=v_r(t)=u_1 * Q_r(t)=\frac{1}{2\pi}\int_{\mathbb{T}} u_1(s)Q_r(t-s)ds=\frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(s)\frac{2\sin(t-s)}{1-2r\cos(t-s)+r^2}ds. \tag{1.9}$$

Obviously, both $P_r(t)$ and $Q_r(t)$ are harmonic in the interior of the unit disk and $Q_r(t)$ is the harmonic conjugate of $P_r(t)$ because they are, respectively, the real and imaginary parts of the analytic kernel $H(z)=H_r(t)=\frac{1+z}{1-z}$.

One should be careful, nevertheless, that the formulas for the kernel functions are only valid for $r<1$ as all these three Fourier multiplier operators degenerate into convolutions with distributions at the boundary ∂D, when $r=1$, that no longer are functions, and this is a crucial element of the theory. For example, the Poisson integral, which for $r<1$ is the Fourier multiplier operator $u_1 \mapsto u_r=\mathcal{F}^{-1}(r^{|n|}\widehat{u_1}(n))$ with multipliers $r^{|n|}$, and therefore can be written as the convolution (1.7) with the well defined harmonic function (1.6), when $r=1$ the multipliers all become 1 and therefore the Poisson kernel degenerates at the boundary ∂D of the disk into a Dirac-δ. Clearly this is consistent with the fact that the Poisson kernel is an approximate identity and that, of course, we want to recover u_1 at the boundary so that the operator should be the identity there. Thus, at $r=1$ we should definitely have $u=u_1*\delta=u_1$. On the other hand, for the conjugate function, given for $r<1$ by the Fourier multiplier operator $u_1 \mapsto v_r=\mathcal{F}^{-1}(-i\,\mathrm{sgn}(n)r^{|n|}\widehat{u_1}(n))$ and also written as the convolution (1.9) with the well defined harmonic function (1.8), when $r=1$ one could be tempted to say that, because we have

$$\lim_{r\to 1} Q_r(t)=\lim_{r\to 1}\frac{2\sin t}{1-2r\cos t+r^2}=\frac{\sin t}{1-\cos t}=\frac{1}{\tan\frac{t}{2}}$$

then, in this case the Fourier multiplier operator $u_1\mapsto v_1=\mathcal{F}^{-1}(-i\,\mathrm{sgn}(n)\widehat{u_1}(n))$, yielding the boundary values of the conjugate function, could actually be written as the convolution of functions

$$v_1(t)=u_1*\frac{1}{\tan\frac{(\cdot)}{2}}(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u_1(s)\frac{1}{\tan\frac{(t-s)}{2}}ds. \tag{1.10}$$

However, a more careful examination of this kernel shows that, in the neighborhood of the origin $\tan(t/2)\sim t/2$ and therefore the kernel for the conjugation operator at $r=1$ would be given by the convolution of u_1 with the very singular $1/\tan(t/2)\sim 2/t$, which is not even Lebesgue integrable in any neighborhood of the origin. It is not even obvious how to define it as a distribution, due to its singular and nonintegrable behavior around the origin, although certainly the Fourier multipliers $-i\,\mathrm{sgn}(n)$ do correspond with some convolution with a distribution kernel. So, in spite of its apparent simplicity, the conjugation kernel is actually more complicated than the Poisson kernel and its Dirac-δ limit at the boundary. In fact, this will be the key to obtaining the convergence of Fourier series in L^p norm, as we will soon see, while the correct definition of the singular convolution integral (1.10) leads to the Hilbert transform and to the modern theory of singular integral operators, around which the Calderon-Zygmund school of harmonic analysis of the second-half of the twentieth century developed.

This whole analysis of holomorphic and harmonic functions on D intentionally started by assuming that the functions were all defined on an open set Ω that contained $\overline{D}$ in order for everything to work perfectly up to, and including, the boundary ∂D. In particular, from the value of the initial harmonic function u at $r=1$ and the absolute convergence of the Taylor series on $\overline{D}$ we guaranteed well defined and absolutely convergent series for the harmonic conjugate v and the full holomorphic function $f=u+iv$, at $r=1$, given by the Fourier multiplier operators

$$u(e^{it})=u_1(t)\mapsto v_1(t)=-i\sum_{n=-\infty}^{\infty}\mathrm{sgn}(n)\widehat{u_1}(n)e^{int} \tag{1.11}$$

and

$$u(e^{it}) = u_1(t) \mapsto f_1(t) = u_1(t) + iv_1(t) = \widehat{u_1}(0) + 2\sum_{n=1}^{\infty} \widehat{u_1}(n)e^{int}, \tag{1.12}$$

even not knowing precisely how to define their convolution distribution kernels, which nevertheless we are sure do exist, from our study of Fourier multiplier operators in the last lesson.

The problem becomes a lot more subtle if we just start with an arbitrary continuous function $g \in C(\mathbb{T})$ or even $g \in L^p(\mathbb{T})$, even complex valued, and try to reproduce the same construction, assuming now that this starting function is the desired boundary value u_1 of our harmonic function u on D, from which we wish to obtain its harmonic conjugate v and finally its boundary value v_1 as well. We will no longer focus on the holomorphic f because its existence and definition is equivalent to that of v, from which it can simply be obtained by $f = u + iv$.

So, the Poisson integral

$$u(z) = u_r(t) = g * P_r(t) = \frac{1}{2\pi}\int_{\mathbb{T}} g(s)P_r(t-s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\frac{1-r^2}{1-2r\cos(t-s)+r^2}ds,$$

is perfectly well defined for $|z| = r < 1$ and defines a harmonic function on D, because $P_r(t)$ is harmonic and one can easily see that $\Delta(g * P_r) = g * \Delta P_r = 0$ by writing the Laplacian in polar coordinates. Besides, from the fact that we already know that the Poisson kernel is an approximate identity, or a summability kernel, we have $g * P_r \to g$ uniformly, i.e. in the supremum norm, as $r \to 1$ when $g \in C(\mathbb{T})$, and in the $L^p(\mathbb{T})$ norm, when $g \in L^p(\mathbb{T})$, for $1 \leq p < \infty$. In this sense, the first part of the problem is solved: we did find a harmonic function u on D whose boundary values, in the sense of these limits as $r \to 1$, is g. In the following lesson we will even see that this solution of the boundary value problem for Laplace's equation on the unit disk D is unique.

As for the harmonic conjugate v we certainly can be sure that

$$v(z) = v_r(t) = g * Q_r(t) = \frac{1}{2\pi}\int_{\mathbb{T}} g(s)Q_r(t-s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\frac{2\sin(t-s)}{1-2r\cos(t-s)+r^2}ds,$$

is also well defined for $|z| = r < 1$, is harmonic and is the only harmonic conjugate of the previous u that vanishes at $z = 0$. It should be noted, though, that in the case that the initial function g is complex, then u is complex and v is a complex harmonic conjugate of u in the sense that the real and imaginary parts are correspondingly conjugate. The crucial problem rests, however, on the boundary values of v as $r \to 1$. It is not at all obvious whether that limit exists pointwise or in norm, and that will be the central issue that we will focus on in the next few lessons: it is called the conjugation problem, and was one of the central questions in complex and harmonic analysis at the beginning of the twentieth century.

To conclude this lesson, it will suffice for now to define the boundary value function of the harmonic conjugate from a Fourier multiplier perspective. In further lessons we will deepen the study of this issue, from different points of view. As we saw before, when assuming that all functions were well defined a priori on $\overline{D}$, the correspondence between the boundary value of u and the boundary value of its harmonic conjugate is given by the Fourier multiplier operator (1.11). Se we have the following definition.

Definition 1.3. *The Fourier multiplier operator $f \in L^1(\mathbb{T}) \mapsto \tilde{f} \in \mathcal{D}'(\mathbb{T})$ defined by*

$$\hat{\tilde{f}}(n) = -i\,\mathrm{sgn}(n)\hat{f}(n),$$

is called the conjugation operator, and $\tilde{f}$ is called the conjugate of f. Also, a space $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, is said to admit conjugation if $\tilde{f} \in L^p(\mathbb{T})$ whenever $f \in L^p(\mathbb{T})$.

As an obvious example, $L^2(\mathbb{T})$ admits conjugation because, as the Fourier multipliers are $\{-i\,\mathrm{sgn}(n)\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ and we know from Theorem 1.9 in the previous lesson that the $L^2(\mathbb{T})$ Fourier multipliers $\mathcal{M}_2$ coincide exactly with $l^\infty(\mathbb{Z})$ then, if $f \in L^2(\mathbb{T})$ the conjugate function will also be $\tilde{f} \in L^2(\mathbb{T})$. Besides

$$\|\tilde{f}\|_{L^2(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})},$$

because we also know, from the same theorem, that the operator norm is equal to the l^∞ norm of the multipliers. Actually, from Plancherel's identity, we can relate the two norms more precisely

$$\|\tilde{f}\|^2_{L^2(\mathbb{T})} = \sum_{-\infty}^{\infty} |\hat{\tilde{f}}(n)|^2 = \sum_{-\infty}^{\infty} |-i\,\mathrm{sgn}(n)\hat{f}(n)|^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 - |\hat{f}(0)|^2 = \|f\|^2_{L^2(\mathbb{T})} - |\hat{f}(0)|^2.$$

Although the definition of an $L^p(\mathbb{T})$ space admitting conjugation only demands that $\tilde{f}$ belong to the same space as f, that is enough to guarantee boundedness of the operator as a consequence of the closed graph theorem, in Functional Analysis (as usual, I recommend Folland's book [1], chapter **5 - Elements of Functional Analysis**, if you need to review some of these fundamental theorems).

Proposition 1.4. *If $L^p(\mathbb{T})$ admits conjugation, then the conjugation operator is bounded on $L^p(\mathbb{T})$.*

Proof. Recall the closed graph theorem that, if X and Y are Banach spaces and a linear map $T : X \to Y$ has a closed graph in the product space $X \times Y$ then the map is bounded $\|Tx\|_Y \leq C\|x\|_X$, for some $C \geq 0$ and all $x \in X$. So, using $X = Y = L^p(\mathbb{T})$, to prove that the graph of the conjugation operator is closed, let us assume that $f_j \to f$ in $L^p(\mathbb{T})$, and $\tilde{f}_j \to g$ in $L^p(\mathbb{T})$. We need only show that $\tilde{f} = g$. But, from the convergence of f_j to f in $L^p(\mathbb{T})$ we know that $\hat{f}_j(n) \to \hat{f}(n)$ uniformly in $n \in \mathbb{Z}$ as $j \to \infty$. And thus $\hat{\tilde{f}}_j(n) = -i\,\mathrm{sgn}(n)\hat{f}_j(n) \to -i\,\mathrm{sgn}(n)\hat{f}(n)$ uniformly in $z \in \mathbb{Z}$ too. On the other hand, as $\tilde{f}_j \to g$ in $L^p(\mathbb{T})$ it must also be that $\hat{\tilde{f}}_j(n) \to \hat{g}(n)$ uniformly in $z \in \mathbb{Z}$. So necessarily

$$\hat{g}(n) = -i\,\mathrm{sgn}(n)\hat{f}(n), \quad \text{for all} \quad n \in \mathbb{Z},$$

which implies that $g = \tilde{f}$. And this concludes the proof. $\square$

A closely related operator, both to the conjugation operator as well as to the operator (1.12), that yields the values at the boundary ∂D of the holomorphic function from the boundary values of its harmonic real part, is the following.

Definition 1.5. *The Fourier multiplier operator $P : L^1(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$ defined, for $f \in L^1(\mathbb{T})$, by*

$$\widehat{Pf}(n) = \begin{cases} \hat{f}(n) & \text{for} \quad n \geq 0, \\ 0 & \text{for} \quad n < 0, \end{cases}$$

is called the Riesz projection operator. In other words, the Riesz projection of f is the distribution with Fourier series $\sum_{n=0}^{\infty} \hat{f}(n)e^{int}$. Also, just like for conjugation, a space $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, is said to admit projection if $Pf \in L^p(\mathbb{T})$ whenever $f \in L^p(\mathbb{T})$.

We then have.

Proposition 1.6. *The Lebesgue space $L^p(\mathbb{T})$, with $1 \leq p \leq \infty$, admits conjugation if and only if it admits projection.*

Proof. The proof is simple and just consists in observing that the Fourier series of Pf can be written in terms of f and $\tilde{f}$, and vice-versa.

So if $L^p(\mathbb{T})$ admits conjugation then, for $f \in L^p(\mathbb{T})$ we have $\tilde{f} \in L^p(\mathbb{T})$ as well. And therefore $\frac{1}{2}\hat{f}(0) + \frac{1}{2}(f + i\tilde{f})$ is also in $L^p(\mathbb{T})$. But this is the Riesz projection of f for its Fourier series is $\sum_{n=0}^{\infty} \hat{f}(n)e^{int}$.

Conversely, if $L^p(\mathbb{T})$ admits projection, then $Pf \in L^p(\mathbb{T})$. And we just need to note that in that case the conjugate $\tilde{f}$ can now be written as $\tilde{f} = -i(2Pf - f - \hat{f}(0))$ which necessarily will also be in $L^p(\mathbb{T})$. $\square$

To conclude this lesson we now present the fundamental theorem that relates the conjugation problem with convergence of Fourier series in $L^p(\mathbb{T})$ norm.

Theorem 1.7. *Fourier series converge in $L^p(\mathbb{T})$ norm for every $f \in L^p(\mathbb{T})$ if and only if $L^p(T)$ admits conjugation.*

Proof. We will combine Theorem 1.1 with the previous proposition to show that $L^p(\mathbb{T})$ admits projection if and only if the norms of the Fourier partial sum operators are uniformly bounded. The main idea is to notice that partial sums of the projection operator are partial sums of the Fourier series shifted in frequency. And shifts in frequency of the whole Fourier series consist of just multiplication by appropriate oscillating exponentials, which do not affect the operator norms. So, essentially, the Fourier partial sum operator norms are uniformly bounded if and only if the correspondingly shifted partial sum operator norms of the projection are also uniformly bounded, and this is equivalent to both converging in $L^p(\mathbb{T})$.

Let us assume first that there exists $C \geq 0$ such that $\|S_N\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \leq C$, for all N. Then, the $2N$ partial sums of the projection can be written as

$$SP_{2N}[f] = \sum_{n=0}^{2N} \hat{f}(n)e^{int} = e^{int}S_N[e^{-int}f], \tag{1.13}$$

and because of the uniform boundedness of the partial sum operators of the Fourier series of f, this also implies now the uniform boundedness of the partial sum operators of the Riesz projection $\|SP_{2N}\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \leq C$. Following then the same idea as in Theorem 1.1 for the partial sums of the Fourier series, we can show here too that $SP_{2N}f$ converges in $L^p(\mathbb{T})$ as a consequence of this uniform boundedness of the partial sum operator norms. In fact, given any $\varepsilon > 0$ we can pick a trigonometric polynomial R for which $\|f - R\|_{L^p(\mathbb{T})} < \varepsilon/2C$ and we then have

$$\|SP_{2N}[f] - SP_{2N}[R]\|_{L^p(\mathbb{T})} = \|SP_{2N}[f-R]\|_{L^p(\mathbb{T})} \leq C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}.$$

Observe now that, because the Projection operator truncates the negative frequencies, its partial sums will never equal an arbitrary trigonometric polynomial, unlike the partial sums of the Fourier series for large enough order. But, in any case, for $N, M > \text{degree}R/2$ then $SP_{2N}[R] = SP_{2M}[R]$ and

$$\begin{aligned}
&\|SP_{2N}[f] - SP_{2M}[f]\|_{L^p(\mathbb{T})} \\
&\quad \leq \|SP_{2N}[f] - SP_{2N}[R]\|_{L^p(\mathbb{T})} + \|SP_{2N}[R] - SP_{2M}[R]\|_{L^p(\mathbb{T})} + \|SP_{2M}[f] - SP_{2M}[R]\|_{L^p(\mathbb{T})} \\
&\quad = \|SP_{2N}[f] - SP_{2N}[R]\|_{L^p(\mathbb{T})} + \|SP_{2M}[f] - SP_{2M}[R]\|_{L^p(\mathbb{T})} < \varepsilon.
\end{aligned}$$

And this we conclude that the partial sums of the Projection operator are Cauchy and therefore converge in $L^p(\mathbb{T})$. Its limit necessarily corresponds is the full Projection operator with Fourier series $\sum_{n=0}^{\infty} \hat{f}(n)e^{int}$.

Now, let us assume conversely that $L^p(\mathbb{T})$ admits projection. Then $f \mapsto Pf$ is a linear bounded operator in $L^p(\mathbb{T})$. And we can write its partial sums of order $2N$ as

$$SP_{2N}[f] = \sum_{n=0}^{2N} \hat{f}(n)e^{int} = Pf - e^{i(2N+1)t}P(e^{-i(2N+1)t}f),$$

so that

$$\|SP_{2N}\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \leq 2\|P\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})},$$

where $\|P\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})}$ here denotes the projection operator norm. The projection partial sum operators are thus uniformly bounded and from (1.13) we conclude that the Fourier partial sum operators are also uniformly bounded, concluding the proof. $\square$

From the examples that we have seen, we could now have concluded that Fourier series converge in $L^2(\mathbb{T})$ because we know $L^2(\mathbb{T})$ admits conjugation.

On the other hand, as we have shown at the beginning of this lesson, after Theorem 1.1, that there are $L^1(\mathbb{T})$ functions whose Fourier series do not converge, we can now conclude that $L^1(\mathbb{T})$ does not admit conjugation. So there must be functions $f \in L^1(\mathbb{T})$ whose conjugate $\tilde{f} \notin L^1(\mathbb{T})$. As a matter of fact, we did see in Lesson 17 the example of the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\cos nt}{\log n} = \sum_{|n|\geq 2} \frac{e^{int}}{2\log|n|},$$

which is a Fourier series of a function in $L^1(\mathbb{T})$, but such that

$$\sum_{n=2}^{\infty} \frac{\sin nt}{\log n} = -i \sum_{|n|\geq 2} \operatorname{sgn}(n) \frac{e^{int}}{2\log|n|}$$

is not. And the latter is precisely the conjugate function of the former, and although it does converge everywhere $t \in \mathbb{T}$, we showed then that it is not a Fourier series.

References

[1] Gerald B. Folland, *Real Analysis, Modern Techniques and Applications,* 2nd Edition, John Wiley & Sons, 1999.
[2] Kenneth Hoffman *Banach Spaces of Analytic Functions,* Dover Publications, 1988.
[3] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 2nd Edition, Dover Publications, 1976.
[4] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 3rd Edition, Cambridge University Press, 2004.